Revisiting the Theory of Finite Size Scaling in Disordered Systems: ν Can Be Less Than 2/d

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For phase transitions in disordered systems, an exact theorem provides a bound on the finite size correlation length exponent: $\nu_{FS} \geq 2/d$. It is believed that the true critical exponent ν of a disorder induced phase transition satisfies the same bound. We argue that in disordered systems the standard averaging introduces a noise, and a corresponding new diverging length scale, characterized by $\nu_{FS} = 2/d$. This length scale, however, is independent of the system's own correlation length ξ. Therefore ν can be less than $2/d$. We illustrate these ideas on two exact examples, with $\nu < 2/d$. We propose a new method of disorder averaging, which achieves a remarkable noise reduction, and thus is able to capture the true exponents.

PACS numbers: 75.10.Nr, 75.40.Mg, 05.70.Fh, 72.15.Rn

Using a very general formulation, Ref. [1] presented an exact theorem, which puts constraints on the finite size correlation length exponent ν_{FS} of a large class of disordered systems: $\nu_{FS} \geq 2/d$, where d is the dimension. This relation is often referred to as the quantum Harris criterion [2]. While many investigations found exponents in accordance with this bound, there is an increasing number of results in contradiction with it. In particular, in a model for charge density waves exact calculations yielded $\nu = 1/2$ below four dimensions [3], and numerical studies on 2d disordered Bose-Hubbard models found $\nu \simeq 0.7$ [4]. Experimentally the Bose glass transition of helium in aerogel [5], and the localization transition in doped semiconductors [6] seem to violate this bound. In this paper we argue that the $\nu_{FS} \geq 2/d$ constraint is characteristic only to the method used to carry out the disorder average, and the true exponent ν is independent of this bound.

To start our considerations of random sytems, we chose the same type of disorder used by Ref. [1]: a binary distribution for, say, a disordered site energy. Typically, physical quantities are calculated by averaging over different disorder realizations. For calculational convenience, the standard method is analogous to the "grand canonical" approach: impurities are put on each site with a given probability, p, and the averaging is carried out for all possible concentration of impurities and their configurations. An alternative method, which could be termed the "canonical" approach, keeps the number of impurities fixed, and the average is taken only over the possible configurations of these impurities. For infinite systems the two methods are equivalent. The density fluctuations in the grand canonical method, however, introduce an extra noise. This noise vanishes in the infinite system, but it may alter the results of the finite size scaling. The "canonical averaging" strongly reduces this noise by excluding density fluctuations.

We now argue that the bound obtained in Ref. [1] is only generated by the noise introduced by the "grand canonical averaging". Different choices, such as using "canonical averaging", produce different bounds. The theorem of Ref. [1] considers a random system where a phase transition is induced by changing the concentration K of site (or bond) impurities. Let Y be any event depending on disorder realizations in a finite volume, with probability $\mathcal{P}(K)$. This $\mathcal{P}(K)$ is calculated by *averaging* over all disordered configurations, and selecting those compatible with Y. Averaging is performed in the "grand canonical" way, since fluctuations in the density of impurities are allowed. From these premises the exact statement $|d\mathcal{P}(K)/dK| \leq$ *const.* $\sqrt{N}$ follows, where N is the system size. A closer look at the proof reveals that this result is derived solely from the concentration fluctuations of the impurities, which were *externally introduced in the averaging process* (see the last equation of the proof in Ref. [1]). Thus the bound on $|d\mathcal{P}(K)/dK|$ does not relate to the intrinsic properties of the system under investigation. It only reflects the "resolution" of the "grand-canonical averaging". In other words, because of the presence of the density fluctuations, the minimal resolvable change in K is $dK \propto 1/\sqrt{N}$. The probability $\mathcal{P}$ can change at most $\mathcal{O}(1)$, immediately explaining the above bound.

On the other hand, if one uses "canonical averaging", then the above inequality does not apply. For, in contrast to the previous case, the number of impurities is now well defined. In the present binary example, the resolvable change of K is bounded only by its minimum allowed increment, $1/N$. Hence, $|d\mathcal{P}(K)/dK| \leq N$. Along the lines of Ref. [1], the inequality $\nu_{FS} \geq 1/d$ now follows. As before, this inequality is characteristic of the "canonical averaging" only, and does not impose any restriction on the true exponent ν of the physical system. The physical reason behind this is that both averaging procedures introduce a *new characteristic length scale*, which has the potential to obscure the true correlation length of the physical system.

It is also important to note that the assumption of a

binary disorder plays a crucial role in deriving the above bounds. For continuous distributions they do not necessarily apply. To see this, consider the following simple example, motivated by the quantum phase transition between the so called Mott-Insulator and Bose-Glass phases, which takes place in interacting bose systems with site disorder. At this transition the renormalization group flows are controlled by a fixed point with *zero* hopping strength [7], thus the system reduces to a collection of independent sites with random energies. Let the distribution of the site energy $\epsilon \in [0, K]$ be

$$P(\epsilon) = \frac{\alpha+1}{K^{\alpha+1}}\big(K-\epsilon\big)^{\alpha}, \tag{1}$$

whith $\alpha > -1$. We generate N independent ϵ_i ($i = 1, \ldots, N$) from the above distribution. We define the finite-size event Y to occur, when *all* ϵ_i's are smaller then a given value $\mu \in (0, K]$. We fix the value of μ, and drive the transition by changing K. As required by the theorem of [1], the probability $\mathcal{P}$ of Y happening is finite at the critical value of the disorder, $K_c = \mu$. It goes to zero exponentially with the system size N for $K > K_c$. Close to the transition, for $\delta = (K - K_c)/K_c \ll 1$, this probability is

$$\mathcal{P}(N, \delta) \simeq e^{-N\delta^{\alpha+1}} \quad . \tag{2}$$

A characteristic length scale ξ_f can be now defined as a function of δ. It is determined from the system size as $N_f = \xi_f^d$, where $\mathcal{P}(N_f, \delta)/\mathcal{P}(N_f, 0) \sim 1/e$. Defining a critical exponent as $\xi_f \propto \delta^{-\nu_{FS}}$ one arrives at $\nu_{FS} = (\alpha+1)/d$. For $\alpha < 1$, ν_{FS} *is less than* $2/d$. While we considered a concrete example, we emphasize that this result can be relevant for *any* transition driven by *local singularities* in the action.

Motivated by the above observations, we now attempt to construct a *modified finite size scaling procedure*, which does have the potential to access the true exponents of the system. The centerpiece of our argument is the following observation. If the distribution of the disorder is given in an analytic form, that uniquely determines K_c, the critical value of the control parameter for the infinite system. However, any *given* disorder realization in a *finite system* could have been generated from disorder distributions with a *range* of parameters, corresponding to a *range* of K_c values. In other words, it is unclear *which* infinite system's finite size realization did one simulate. A distribution is characterized completely by its moments. Typically, K_c is linked to some of these moments, for instance the dispersion. For a finite system of size N, this dispersion is determined only with a relative uncertainty of $\mathcal{O}(1/\sqrt{N})$. Therefore there is a range of distribution parameters which are compatible with the specific realization, and thus could have generated it. This raises the problem, *which* K_c *to use* in a finite size scaling analysis.

The standard procedure answers this question by assuming that one can use a single K_c for all samples generated from the same distribution. However, the above argument suggests that the very same sample may be the realization of distributions with different parameters, leading to an inherent noise in the procedure, similar to the above considered binary examples. In order to avoid such a built-in noise, we now propose a modified finite size scaling procedure for disordered systems. We suggest that for *each disorder realization* one should identify the distribution, and in particular the critical parameter K_c^r, which it *most likely* corresponds to. In practice this might be difficult, and we return to this question later. For the moment, we only assume that it is possible to identify K_c^r. We propose that the natural control parameter of the critical behaviour is $\Delta = (K - K_c^r)/K_c^r$. The act of averaging should then be performed for the samples with the same Δ. We propose to adopt the finite size scaling hypothesis for the critical behaviour of a generic physical quantity Q,

$$\bar{Q}(L, \Delta) = L^{-y} q(L\Delta^{\nu}) \quad , \tag{3}$$

where $q(z)$ is a universal scaling function, and y, ν are the critical exponents for Q, and the true correlation length $\xi \propto \Delta^{-\nu}$. Note that some aspects of this proposition are already practiced in numerical studies: sizeable noise-reduction is customarily reached by adjusting the random variables *after* they are generated, e.g. in order to keep their mean value constant.

Next we assume the validity of Eq.3 and perform the standard finite size scaling, to demonstrate how that procedure's inherent noise can mask the true critical behaviour. Some of the key results of the analysis are: *i*) we find that the exponent of the intrinsic correlation length ν might be different from ν_{FS} appearing in the standard finite size scaling. Therefore the theorem of Ref. [1] does not provide constraints on the true exponent ν. *ii*) In particular, ν *can be* less than $2/d$. In this case typically $\nu_{FS} = 2/d$.

The standard finite size scaling procedure [8] in disordered systems calls for calculating a physical quantity, Q, such as the critical susceptibility, for different values of N and K, the system size and control parameter, each time performing the calculations for a number of disorder realizations. Averaging over the disorder yields $\langle Q(K)\rangle$, and the critical coupling K_c is then identified for instance from a crossing pattern [9]. Requiring the collapse of the data, when plotted as a function of $L^{1/\nu}\delta$, where $\delta = (K - K_c)/K_c$, determines the exponents.

To make contact between the standard scaling procedure and Eq.3, a relation between the unique K_c and the fluctuating K_c^r has to be constructed. A simple representation of the inherent noise, or uncertainity, is to assume the validity of the central limit theorem for K_c^r

$$\Delta = \delta + \frac{D}{L^{d/2}} x \quad , \tag{4}$$

where x is a random variable with a distribution width of $\mathcal{O}(1)$. Here D measures the scatter in K_c^r, and δ is the distance from the average critical point K_c. As we have seen, this is not necessarily true for all systems, and we will return to the case when the fluctuations scale with a different power.

The standard procedure neglects the fluctuations of K_c^r, which is equivalent to averaging $\bar{Q}$ over the random variable x of Eq.4:

$$\langle Q\rangle = L^{-y}\left\langle q\left(D^\nu L^{1-\frac{d\nu}{2}}\left(x+\frac{\delta L^{d/2}}{D}\right)^\nu\right)\right\rangle \quad . \tag{5}$$

Here the x average is denoted by $\langle\ldots\rangle$, corresponding to the standard averaging procedure, as opposed to $\bar{Q}$, the correlated averaging of the new procedure in Eq.3.

First we analyze the critical point itself, then we shall proceed to extract the critical behaviour of the correlation length. At $\delta = 0$ the scaling form for Q is

$$\langle Q\rangle = L^{-y}\left\langle q\left(D^\nu x^\nu L^{1-\frac{d\nu}{2}}\right)\right\rangle \quad . \tag{6}$$

For $\nu > 2/d$ the argument of the scaling function approaches zero with increasing system size, and the L dependence of the *averaged* quantity $\langle Q(L)\rangle$ is characterized by the *intrinsic* exponent y. Here we use the customary assumption that the universal scaling function $q(z)$ approaches a finite value as $z \to 0$.

In the $\nu < 2/d$ case, however, the argument of $q(z)$ goes to large values, probing deeply non-critical regions, even though the system is assumed to be *at criticality*. To highlight the consequences of this, we proceed with a generic form for the asymptotic behaviour of the scaling function, adopting $q(z) \propto z^{-\beta}$. From Eq.6 $\langle Q\rangle \propto L^{-\gamma}$, where $\gamma = y + \beta(1 - d\nu/2)$. Clearly the L dependence of the averaged $\langle Q\rangle$ is *different* from the intrinsic value y.

Next we develop an understanding of the region in the proximity of the critical point, i.e. the case of finite δ. Let us first focus on $\nu < 2/d$. From Eq.5 one identifies two scaling regions, governed by *two different* characteristic diverging length scales.

For large system sizes inevitably $D^\nu L^{1-d\nu/2} \gg 1$, so the argument of $q(z)$ again extends to large values. Utilizing the previous asymptotic model form,

$$\langle Q\rangle = L^{-\gamma}\hat{q}\left(\delta L^{d/2}\right) \quad . \tag{7}$$

from which a length scale can be identified, characterizing the finite size scaling of $\langle Q\rangle$, averaged in the standard way. It diverges with an exponent $\nu_{FS} = 2/d$ *even though the true exponent ν is less than* $2/d$. This result now demonstrates in general, what has been observed earlier for the binary example: the standard, or "grand canonical" averaging introduces a noise, which in turn generates a new length scale and a corresponding new exponent into the analysis.

The other scaling region is reached when $\delta L^{d/2}/D \gg 1$. In this limit

$$\langle Q\rangle = L^{-y}q\left(\delta^\nu L\right) \quad . \tag{8}$$

As is known, for large values of $\delta^\nu L$, the ν exponent is not accessible by finite size scaling [8], hence $\delta^\nu L$ should be kept around unity. Therefore the determination of ν requires the study of the region *away from the asymptotics*: large δ and small system sizes. For weak disorder ($D \ll 1$) this window in fact might be wide enough for practical purposes. To reiterate, however, studies concentrating on the asymptotic region are bound to see $\nu_{FS} = 2/d$.

In the case of $\nu > 2/d$ the standard procedure is capable of accessing the true ν: it can be obtained from $\langle Q\rangle$ by increasing the system size to the extent of $\delta L^{d/2}/D \gg 1$, but keeping $\delta^\nu L \propto \mathcal{O}(1)$. This again implies avoiding the "non-scaling" region around $\delta = 0$. For strong disorder and small available system sizes, one can end up again with large arguments of $q(z)$, and consequently in the scaling regime described by ν_{FS} and γ (Eq.7). There are several additional crossover regimes in the parameter space, which can be studied based on Eq.5.

What happens if instead of the central-limit-theorem form, $L^{-d/2}$, the fluctuations of K_c^r are described by some other power law? For instance, on physical grounds, the fluctuations may scale with the correlation length exponent

$$\Delta = \delta + \frac{D}{L^{1/\nu}}x \quad . \tag{9}$$

Substituting this expression into Eq.3 and averaging over x shows that the standard procedure and our proposition give the same result for the exponents, although the scaling function changes due to the difference in averaging.

Equation 9 can help incorporate the idea of K_c^r in the scaling. One can appreciate that from looking at a finite sample, it is far from trivial to identify K_c^r, belonging to the infinite system. A solution might be suggested by recalling that for ordered classical magnets, the maximum of the susceptibility of a finite size sample is shifted as: $T_c(L) - T_c(\infty) \propto L^{-1/\nu}$, just as in Eq.9. Scaling then can be performed in terms of $T_c(L)$, resulting in the correct exponents. Even in the absence of knowledge of the K_c^r of the corresponding infinite system, one can still extract a $\tilde{K}_c^r$ from a specific feature of a critical quantity of the *finite size system*. Using this $\tilde{K}_c^r$ in our new scaling approach should provide the correct exponent ν, provided that $\tilde{K}_c^r - K_c^r \propto L^{-1/\nu}$, a reasonable assumption.

We are thus left with the task of identifying $\tilde{K}_c^r$ of a finite system. For many quantum systems at $T = 0$ a reasonable proposition for $\tilde{K}_c^r$ might be the value of K, where the gap to the first excitation vanishes or has a minimum. For classical systems $\tilde{K}_c^r$ may be identified where some critical susceptibility exhibits a maximum.

To demonstrate the above ideas, consider strongly interacting bosons in a random potential at zero temperature. In Ref. [7] renormalization flows were generated by integrating out the sites with highest excitation energies. For infinite range hopping the renormalization group (RG) equations are *exact.* In particular, at the Mott-Insulator to Superfluid transition weak disorder is irrelevant and $\nu = 1/d$. Because of the presence of an underlying RG one expects the validity of finite-size scaling.

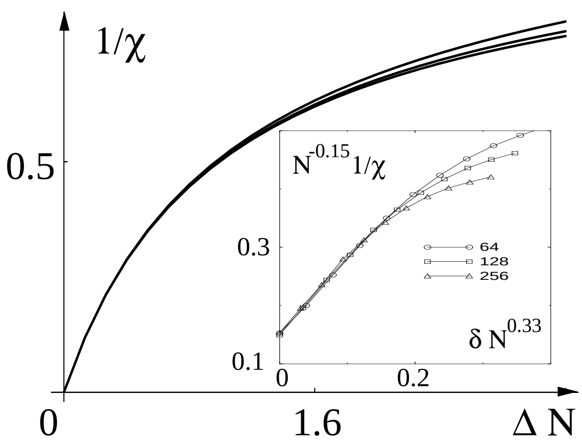

FIG. 1. Scaling plot of the inverse susceptibility using the novel and the standard (insert) averaging procedure for system sizes N=64,128,256.

We carried out the finite-size scaling analysis of the average local susceptibility at weak but finite disorder for system sizes $N = 64, 128, 256$. First we used the standard averaging procedure (insert of Fig.1), and we obtained $\nu_{FS} \simeq 3/d$ after averaging over 1024 realizations of a uniform disorder distribution of the random potential. The collapse of the curves for different system sizes is not perfect, and we expect that as N increases, $\nu_{FS} \to 2/d$. Fig.1 shows the same quantity scaled by using $\tilde{K}_c^r$ extracted from the divergence of the susceptibility for each sample separately. The scaling is convincing, and yields the exact exponent $\nu = 1/d$. The exhibited curves were obtained by averaging over much fewer samples than before, only 16, yet the scaling region extends by more than an order of magnitude further in terms of the scaling variable, $N\Delta$, clearly demonstrating a very effective noise reduction.

In some numerical studies, such as in Ref. [4] a $\nu_{FS} < 2/d$ has been reported, using the traditional averaging procedure. We would like to emphasize that this finding can be perfectly accomodated in the present theory. First, our analysis does *not* suggest that ν_{FS} *must* be greater or equal to $2/d$: this is only the most likely scenario. If, for instance, Eq.9 describes the fluctuations of K_c^r, then $\nu_{FS} = \nu$, and thus can be less than $2/d$. Apparently, this is the case in the example of the Mott-Insulator to Bose-Glass transition in Eq.2. Second, as emphasized after Eq.8, if the fluctuations of K_c^r are small and the sample size is not too big, then the true ν can and will be observed in finite size scaling. Finally, this theory is *not addressing* the problems associated with distributions with long power-law tails [10], or multicritical fixed points [11]. After averaging, we still expect a gaussian distribution, with exponential tails.

Now we would like to reflect on the Harris criterion [2]. An insightful derivation imagines dividing the sample to blocks of size ξ. For $\nu < 2/d$, the fluctuations of the local "T_c's" of the blocks are bigger than the distance from the true T_c and it is concluded that ν cannot be smaller than $2/d$. In our framework this negative result only means that the blocking procedure ceases to be a valid approach to the infinite system, and says nothing about the value of the inherent exponents. An RG based investigation of this problem will be given elsewhere.

In sum, we reinvestigated the theory of finite size scaling in disordered systems. We found that the standard averaging procedure introduces a new diverging length scale into the problem, therefore the finite size scaling exponent ν_{FS} may be unrelated to the inherent ν of the true correlation length. In particular, we constructed two examples explicitly, where *exact* calculations proved that the inherent ν is less than $2/d$. We proposed an alternative method, which achieves a remarkable noise reduction, and therefore is capable of accessing the true exponents of the physical problem.

We acknowledge useful discussions with L. Chayes, T. Giamarchi, A. Otterlo, H. Rieger, S. Sachdev, R. Singh, P. Weichman, and P. Young. This research was supported by NSF DMR-95-28535.

[1] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).
[2] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[3] O. Narayan and D.S. Fisher, Phys. Rev. **B46**, 11520 (1992).
[4] J. Kisker and H. Rieger, cond-mat/9612027.
[5] P. Crowell, F. Van Keuls and J. Reppy, cond-mat/9612133.
[6] M. Paalanen, T. Rosenbaum, G. Thomas, and R. Bhatt, Phys. Rev. Lett. **48** 1284 (1982); H. Stupp et al, Phys. Rev. Lett. **71**, 2634 (1993).
[7] F. Pázmándi and G. T. Zimányi, cond-mat/9702169
[8] See the papers in *Finite Size Scaling*, ed. J.L. Cardy, North-Holland (1988).
[9] K. Binder, Z. Phys. **B43**, 119 (1981).
[10] D. Fisher, Phys. Rev. **B 51**, 6411 (1995).
[11] R. Singh and M. Fisher, Phys. Rev. Lett. **60**, 548 (1988).